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A PREDICTION METHOD FOR AERODYNAMIC SOUND PRODUCED BY CLOSELY SPACED ELEMENTS IN AIR DUCTS

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1. INTRODUCTION

The accurate prediction of aerodynamic sound production in air ducts at the design stage is of engineering importance since it is almost impossible to remedy the flow-generated noise problems after the installation of the ventilation systems. Over the years, a number of investigators [1–6] have tried to devise a prediction technique for aerodynamic sound in air ducts. Nevertheless, their predictive technique can only be applied to an isolated element in an air duct. Current design methods such as CIBSE guide [7] and ASHRARE handbook [8] are based upon the work of several investigators on aerodynamic noise produced by an isolated element in low speed flow ducts. However, some in-duct elements or duct discontinuities are very close to each other and the noise generated by them is very different from that of an isolated element. In practice, some ventilation designers use the current design methods directly to predict the flow-generated noise produced by closely spaced elements though they are designed for isolated elements in air ducts. The specific problem that motivated this study is the inaccurate noise prediction due to the interaction between closely spaced in-duct elements. The aim of the present investigation is therefore to predict sound level and spectral content of the noise radiated by the closely spaced spoilers in low speed air flow ducts.

2. SOUND POWER RADIATED BY TWO CLOSELY SPACED SPOILERS IN AIR DUCT

2.1. INTERACTION OF TWO NOISE SOURCES

It is straightforward to obtain the acoustic power spectral densities $W(\omega)$ from reference [9] as follows:

$$W(\omega) = \sum_{mn} \frac{\rho_0 c_{mn}}{4A} \lim_{T\to\infty} \frac{\pi}{T} |Q_{mn,\omega}(\omega)|^2, \tag{1}$$

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where A is the cross-section area of the duct, ρ_0 is the ambient density of air, c_{mn} is the mode axial phase speed, and the source volume integral $Q_{mn}(\omega)$ is given by

$$Q_{mn}(\omega) = \frac{1}{c_{mn}} \iint_s f_3(x', y') \psi^*_{mn}(x', y') \mathrm{e}^{ik_{mn}z^*} \,\mathrm{d}x' \,\mathrm{d}y', \tag{2}$$

where ψ_{mn} is the normalized mode cross-section function, k_{mn} is the mode axial wave number and $f_3(x', y')$ is the force per unit density acting per unit area over the cross-section $z' = z''$ occupied by the thin rectangular spoiler, the force arising from the difference in fluctuating pressure acting on the two sides of the spoiler. These fluctuations in pressure are caused mainly by the formation and convection of eddies as the oncoming flow separates at the edge of the spoiler. (x', y', z') is the co-ordinate of the spoiler.

The sound power radiated by two sources in the duct can be written as

$$W(z, t) = \iint_s \bar{I}(x, y) \,\mathrm{d}x \,\mathrm{d}y = \iint_s \tfrac{1}{2} \mathrm{Re}\{p_{1+2} u^*_{1+2}\} \,\mathrm{d}x \,\mathrm{d}y \tag{3}$$

or, in dimensionless form,

$$\frac{W(z, t)}{A\rho_0 c_0^3} = \sum_{m,n} \left\{ \begin{matrix} \pi^1_{mn}(z, t)\mu^{1*}_{mn}(z, t) + \pi^1_{mn}(z, t)\mu^{2*}_{mn}(z, t) \\ + \pi^2_{mn}(z, t)\mu^{1*}_{mn}(z, t) + \pi^2_{mn}(z, t)\mu^{2*}_{mn}(z, t) \end{matrix} \right\}, \tag{4}$$

where π_{mn} and μ_{mn} are the dimensional modal pressure ratio and the modal axial particle velocity ratio respectively. When the first and the second sources are located at $z = 0$ and $z = d$, respectively, the power spectral density of the sound field in duct can therefore be written as

$$\bar{W}(\omega) = \sum_{mn} \frac{1}{4A} \rho_0 c_{mn} \{ S^{11}_{mn}(\omega) + [S^{21}_{mn}(\omega) + S^{12}_{mn}(\omega)] \cos(k_{mn} d) + S^{22}_{mn}(\omega) \}, \tag{5}$$

where all quantities $S_{mn}(\omega)$ represent the power spectral density of the corresponding source volume integral $Q_{mn}(\omega)$, as defined by

$$S^{11}_{mn}(\omega) = \lim_{T\to\infty} \frac{\pi}{T} |Q^1_{mn}(\omega)|^2, \tag{6}$$

$$S^{12}_{mn}(\omega) = S^{21^*}_{mn}(\omega) = \lim_{T\to\infty} \frac{\pi}{T} (Q^1_{mn} Q^{2*}_{mn}), \tag{7}$$

$$S^{22}_{mn}(\omega) = \lim_{T\to\infty} \frac{\pi}{T} |Q^2_{mn}(\omega)|^2. \tag{8}$$

It is noted that $|S_{mn}^{12}(\omega)|^2 = S_{mn}^{11}(\omega)S_{mn}^{22}(\omega)$, which is the condition of coherence, i.e., $\gamma_{iij}^2 = |S_{ij}|^2/S_{ii}S_{jj} = 1$. It can be verified by the summation of the powers radiated by the primary and second sources.

2.2. THE DISCUSSION OF SOLUTION FOR SOUND PRODUCED BY TWO CLOSELY SPACED SPOILERS IN AN INFINITE HARD WALLED DUCT

In reference [5], the modal quantity $S_{mn}(\omega)$ is related to the power spectrum $S_F(\omega)$ of the total drag force on the thin rectangular spoiler, and the resulting expression for the spectral density $S_{mn}(\omega)$ is

$$S_{mn}(\omega) = \frac{1}{\rho_0^2 |c_{mn}|^2} S_F(\omega) \frac{1}{A_s} \iint_{A_s} |\psi_{mn}(x_k)|^2 \, \mathrm{d}s(x_k). \tag{9}$$

Then,

$$S_{mn}^{11}(\omega) = \frac{1}{\rho_0^2 |c_{mn}|^2} S_{F_1}(\omega) \frac{1}{A_{s1}} \iint_{A_{s1}} |\psi_{mn}(x_{k1})|^2 \, \mathrm{d}s(x_{k1}), \tag{10}$$

$$S_{mn}^{22}(\omega) = \frac{1}{\rho_0^2 |c_{mn}|^2} S_{F_2}(\omega) \frac{1}{A_{s2}} \iint_{A_{s2}} |\psi_{mn}(x_{k2})|^2 \, \mathrm{d}s(x_{k2}), \tag{11}$$

$$|S_{mn}^{12}(\omega)|^2 = |S_{mn}^{21}(\omega)|^2 = S_{mn}^{11} S_{mn}^{22}. \tag{12}$$

It is defined that $S_{mn}^{12} \equiv |S_{mn}^{12}| \mathrm{e}^{\mathrm{i}\phi(\omega)}$ and equation (5) can therefore be written as

$$f(S_{mn}^{11}) = W_{11}(\omega) \equiv \frac{1}{4A\rho_0} S_{F_1}(\omega) \sum_{mn} \frac{1}{|c_{mn}| A_{s1}} \iint_{A_{s1}} |\psi_{mn}(x_{k1})|^2 \, \mathrm{d}s(x_{k1}), \tag{13}$$

$$f(S_{mn}^{22}) = W_{22}(\omega) \equiv \frac{1}{4A\rho_0} S_{F_2}(\omega) \sum_{mn} \frac{1}{|c_{mn}| A_{s2}} \iint_{A_{s2}} |\psi_{mn}(x_{k2})|^2 \, \mathrm{d}s(x_{k2}), \tag{14}$$

$$f(S_{mn}^{12}, S_{mn}^{21}) = W_{12}(\omega) \equiv \frac{1}{4A\rho_0} \sqrt{S_{F_1} S_{F_2}} \sum_{mn} \frac{1}{|c_{mn}|\sqrt{A_{s1} A_{s2}}} \sqrt{\iint_{A_{s1}} |\psi_{mn}(x_{k1})|^2 \, \mathrm{d}s(x_{k1})}$$
$$\sqrt{\iint_{A_{s2}} |\psi_{mn}(x_{k2})|^2 \, \mathrm{d}s(x_{k2})}\; 2\cos[\phi(\omega)] \cos(k_{mn} d). \tag{15}$$

The summation of $(1/c_{mn})$ and the summation of $[1/c_{mn} \cos(k_{mn} d)]$ for all propagating modes need to be solved so that the above equations can be applied in this present investigation for practical engineering need. It can be noted that the ratios (c_0/c_{mn}) for a given mode can be expressed in terms of the integers m and n, i.e.,

$$c_0/c_{mn} = \sqrt{1 - (m\pi/ka)^2 - (n\pi/kb)^2}. \tag{16}$$

If m and n are now regarded as continuous variables, the ratio (c_0/c_{mn}) can be thought of as another continuous variable, which is a function of m and n, i.e. $(c_0/c_{mn}) \approx f(m, n)$. It can be estimated that

$$\sum_{m,n}^{N} \frac{1}{c_{mn}} = \frac{1}{c_0} \sum_{m,n}^{N} \frac{c_0}{c_{mn}} \approx \frac{1}{c_0} \int_0^{kb/\pi} \int_0^{ka/\pi} f(m, n)\, \mathrm{d}m\, \mathrm{d}n, \tag{17}$$

where (ka/π) and (kb/π) are the maximum values of the continuous variables m and n for which modes propagate at the frequency in question. According to the concept in reference [5], the following summation is

$$\sum_{m,n}^{N} \frac{c_0}{c_{mn}} = \frac{k^2 ab}{6\pi} + \frac{k}{8}(a + b). \tag{18}$$

Referring to Appendix A, the sum of $\sum_{m,n}^{N} (c_0/c_{mn}) \cos(k_{mn} d)$ can be obtained as follows:

$$\sum_{m,n}^{N} \left(\frac{c_0}{c_{mn}}\right) \cos(k_{mn} d) \approx \iint_s \sqrt{1 - \left(\frac{m\pi}{ka}\right)^2 - \left(\frac{n\pi}{kb}\right)^2} \cos\left[(kd)\cdot \sqrt{1 - \left(\frac{m\pi}{ka}\right)^2 - \left(\frac{n\pi}{kb}\right)^2}\right] \mathrm{d}m\, \mathrm{d}n$$

$$= \frac{k^2 ab}{2\pi}\left[\frac{\sin e}{e} + \frac{2\cos e}{e^2} - \frac{2\sin e}{e^3}\right] + \frac{k(a+b)}{4}\left[J_0(e) - \frac{J_1(e)}{e}\right], \tag{19}$$

where $e = kd$, and J_0 and J_1 are the Bessel functions.

In addition,

$$\frac{1}{A_{s1}} \iint_{A_{s1}} |\psi_{mn}(x_{k1})|^2\, \mathrm{d}s(x_{k1}) = \left[1 + \frac{a}{m\pi d_1} \sin\left(\frac{m\pi d_1}{a}\right) \cos\left(\frac{2m\pi \bar{a}_1}{a}\right)\right]$$

$$\times \left[1 + \frac{b}{m\pi h_1} \sin\left(\frac{m\pi h_1}{b}\right) \cos\left(\frac{2m\pi \bar{b}_1}{b}\right)\right] \approx 1, \tag{20}$$

where $(\bar{a}_1, \bar{b}_1)$ is the spoiler 1 central co-ordinate, d_1, h_1 are its length and width respectively. Similarly,

$$\frac{1}{A_{s2}} \iint_{A_{s2}} |\psi_{mn}(x_{k2})|^2\, \mathrm{d}s(x_{k2}) \approx 1. \tag{21}$$

Equations (13)–(15) can then be rewritten as

$$W_{11}(\omega) = \frac{\omega^2}{24\pi\rho_0 c_0^3}\left[1 + \frac{3\pi c_0}{4\omega}\frac{(a+b)}{A}\right] S_{F_1}(\omega),$$

$$W_{22}(\omega) = \frac{\omega^2}{24\pi\rho_0 c_0^3}\left[1 + \frac{3\pi c_0}{4\omega}\frac{(a+b)}{A}\right] S_{F_2}(\omega),$$

$$W_{12}(\omega) = \frac{\omega^2}{24\pi\rho_0 c_0^3}\left[1 + \frac{3\pi c_0}{4\omega}\frac{(a+b)}{A}\right] Q 2\cos[\phi(\omega)]\sqrt{S_{F_1}(\omega) S_{F_2}(\omega)}, \quad \text{(22a, b, c)}$$

where Q is given by

$$Q = \frac{k^2 ab/4\pi 2[\sin e/e + 2\cos e/e^2 - 2\sin e/e^3] + k(a+b)/8 2[J_0(e) - J_1(e)/e]}{[k^2 ab/4\pi + k(a+b)/8]}$$

and $e = kd$. (23)

The total power spectral density in the duct for the case of $f > f_0$ is

$$W(\omega) = W_{11} + W_{22} + W_{12}$$

$$= \frac{\omega^2}{24\pi\rho_0 c_0^3}\left[1 + \frac{3\pi c_0}{4\omega}\frac{(a+b)}{A}\right]\{S_{F_1} + S_{F_2} + 2Q\cos[\phi(\omega)]\sqrt{S_{F_1} S_{F_2}}\}. \quad (24)$$

For $f < f_0$. Equations (13)–(15) can be reduced to a simple expression for plane wave propagation defined by $m, n = 0$. The total power spectral density is

$$W(\omega) = \frac{1}{4A\rho_0 c_0}\{S_{F_1} + S_{F_2} + \cos(kd) 2\cos[\phi(\omega)]\sqrt{S_{F_1} S_{F_2}}\}. \quad (25)$$

Thus, two very simple results have been derived for the source power radiated by two assumed form of source distribution, under the conditions of plane wave and multi-modal sound propagation in an infinite duct.

3. SCALING A LAW FOR PRACTICAL ENGINEERING

3.1. THE RELATIONSHIP BETWEEN FLUCTUATING AND STEADY STATE DRAG FORCES

In the development of the theory, it is assumed that the root mean square (r.m.s.) fluctuating drag force acting on the spoiler is directly proportional to the steady state drag force $\bar{F}_z$. This assumption was also used by Gordon [2, 3], Nelson and Morfey [5] and is supported by the experiment of Heller and Widnall [4]. If the proportional frequency band defined by the limits $(f_c/\alpha, f_c\alpha)$ where f_c is the centre frequency, is considered, the ratio of the r.m.s. fluctuating drag force to the steady state drag force is dependent only on the Strouhal number only. This can be expressed as

$$(F_z)_{r.m.s.} = K(\mathrm{St})\bar{F}_z \tag{26}$$

(band f_c/α to $f_c\alpha$) where the numerical factor $K(\mathrm{St})$ depends on the choice of α, the Strouhal number St is given by $\mathrm{St} = f_c r/U_c$, where U_c is the flow velocity in the constriction provided by the spoiler (U_c is defined by the volume flow rate q and the area of the duct constriction A_c, such that $U_c = q/A_c$) and r is a characteristic dimension, and the steady state drag force acting on the spoiler can be written in terms of a drag coefficient C_D and is expressed as

$$\bar{F}_z \equiv C_D(\tfrac{1}{2}\rho_0 U_\infty^2)A_s = C_D(\tfrac{1}{2}\rho_0 U_c^2)\sigma^2(1-\sigma)A, \tag{27}$$

where σ is the open area ratio, A is the area of duct cross-section, A_s is the face area of the flat spoiler and U_∞ is the duct velocity q/A; and it is assumed that steady state drag forces acting on the two spoilers are the same, i.e., $\bar{F}_{z1} = \bar{F}_{z2}$.

3.2. THE RELATIONSHIP BETWEEN SOUND POWER AND FLOW PARAMETERS

Using previously derived equations (24) and (25) for the sound power transmitted along the duct under multi-mode and plane wave radiation conditions, the sound power radiated in a given bandwidth can be expressed as follows, for frequencies above and below the cut-on frequency of the lowest transverse duct mode.

The mean square value of the fluctuating force in a given band is given by

$$(\bar{F}_z^2)_{\Delta F} = \int_{\omega_1}^{\omega_2} S_F(\omega)\,\mathrm{d}\omega, \tag{28}$$

where $S_F(\omega)$ is the power spectrum of the total drag force on the spoiler.

Using equations (26) and (27), the sound power can be expressed in terms of drag coefficient. After some algebraic manipulation, it can be shown that when the two spoilers have the same shape,

$f_c < f_0$,

$$W_{\Delta F} \approx (1/4A\rho_0 c_0)\{(\bar{F}_{z1}^2)_{\Delta F} + (\bar{F}_{z2}^2)_{\Delta F} + 2\cos(kd)\cos[\phi(\omega)]\sqrt{(\bar{F}_{z1}^2)_{\Delta F}\,(\bar{F}_{z1}^2)_{\Delta F}}\}$$

$$= (\rho_0/16c_0)AK^2(\mathrm{St})[\sigma^2(1-\sigma)]^2 C_D^2 U_c^4\{2 + 2\cos(kd)\cos[\phi(\omega)]\}, \tag{29}$$

$f_c > f_0$

$$W_{\Delta F} \approx (\omega_c^2/24\pi\rho_0 c_0^3)[1 + (3\pi c_0/4\omega_c)(a+b)/A]\{(\bar{F}_{z1}^2)_{\Delta F} + (\bar{F}_{z2}^2)_{\Delta F}$$

$$+ 2Q\cos[\phi(\omega)]\sqrt{(\bar{F}_{z1}^2)_{\Delta F}\,(\bar{F}_{z1}^2)_{\Delta F}}\}$$

$$= (\rho_0\pi/24c_0^3)[1 + 3\pi c_0/4\omega_c)(a + b)/A](A/r)^2(\mathrm{St})^2 K^2(\mathrm{St})$$

$$\times [\sigma^2(1 - \sigma)]^2 C_D^2 U_c^6 \{2 + 2Q\cos[\phi(\omega)]\}, \qquad (30)$$

where a, b are the duct cross-section dimensions and d is the distance between two spoilers. Since the sound power measurements are made in proportional frequency bands, the above scaling laws may be used as they stand to normalize an experimental data. The inferred infinite-duct values of radiated sound power level SWL_D in the frequency band can thus be normalized by evaluating

$f_c < f_0$

$$120 + 20\log_{10}K(\mathrm{St}) = \mathrm{SWL}_D - 10\log_{10}\{\rho_0 A[\sigma^2(1-\sigma)]^2 C_D^2 U_c^4/16c_0\}$$

$$- 10\log_{10}[2 + 2\cos(kd)], \qquad (31)$$

$f_c > f_0$

$$120 + 20\log_{10}K(\mathrm{St}) = \mathrm{SWL}_D - 10\log_{10}\{\rho_0\pi A^2(\mathrm{St})^2[\sigma^2(1-\sigma)]^2 C_D^2 U_c^6/24c_0^3 r^2\}$$

$$- 10\log_{10}[1 + (3\pi c_0/4\omega_c)(a + b)/A] - 10\log_{10}\{2 + 2Q\}. \qquad (32)$$

It is assumed in the above equations that $S_{F_1} = S_{F_2}$, i.e., $\phi(\omega) = 0$. It should be noted that when the two sources are located at the same position, the total sound power should be the same as that of a single source.

It can be seen that these predictive equations (equations (31) and (32)) are similar to those derived by Nelson and Morfey in reference [5] and can be expressed in terms of easily measurable engineering parameters, together with the single Strouhal number-dependent constant $K(\mathrm{St})$. The additional terms in equations (31) and (32) as they are compared to equations of Nelson and Morfey represent the interaction between two aerodynamic noise sources. Nelson and Morfey measured the sound power levels radiated by different flat plate flow spoilers in a rectangular air duct at various velocities and finally they produced a normalized spectrum [5]. It should be noted that 6 dB corrections need to be subtracted from the values of $K^2(\mathrm{St})$ in their normalized spectrum [10]. Together with the corrected normalized spectrum $K^2(\mathrm{St})$ of Nelson and Morfey, the inferred-duct values of sound power level in $\frac{1}{3}$ octave bands radiated by two closely spaced spoilers can be obtained by equations (31) and (32). The predictive equations (31) and (32) can therefore form a basis of a generalized prediction method for aerodynamic sound generated by closely spaced induct elements. However, the validity of these predictive equations should be checked against experimental data. This will certainly require further work.

4. CONCLUSIONS

The predictive equations developed here which permit the determination of sound power radiated by two closely spaced spoilers only require simple flow parameters and corrected normalized spectrum $K^2(\mathrm{St})$ derived by Nelson and Morfey. It needs to be seen whether this predictive technique based on their corrected normalized spectrum $K^2(\mathrm{St})$ can be applied to a wider range of flow duct discontinuities. The ultimate objective of the present study is therefore to extend this method to predict the sound level and spectral distribution of the additional acoustic energy produced by the combination of any given duct discontinuities in any given duct air flow velocities. This study certainly provides a basis for further investigation.

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APPENDIX A: CALCULATION OF THE SUMMATION IN EQUATION (19)

According to the concept of reference [5], the double integration expressed in equation (19) evaluates the volume enclosed by the surface $(c_0/c_{mn})\cos k_{mn}d$ for all propagating modes.

Let $x = \pi/ka\,m$, $y = \pi/kb\,n$. Then equation (19) can be written as

$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\cos\left(kd\sqrt{1-x^2-y^2}\right)\left(\frac{ka}{\pi}\right)\left(\frac{kb}{\pi}\right)\mathrm{d}x\,\mathrm{d}y$$

$$= \left(\frac{ka}{\pi}\right)\left(\frac{kb}{\pi}\right)\int_0^1 \left\{\int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\cos(kd\sqrt{1-x^2-y^2})\,\mathrm{d}x\right\}\mathrm{d}y, \tag{A1}$$

let $x_1 = \sqrt{1-x^2-y^2/1-y^2}$, $e = kd$; then $\mathrm{d}x = x_1\sqrt{1-y^2/1-x_1^2}$. Equation (A1) can be simplified as

$$I = \left(\frac{ka}{\pi}\right)\left(\frac{kb}{\pi}\right)\int_0^1 (1-y^2)\left\{\int_0^1 \frac{x_1^2}{\sqrt{1-x_1^2}}\cos\,(e\sqrt{1-y^2}\,x_1)\,\mathrm{d}x_1\right\}\mathrm{d}y. \tag{A2}$$

On the right of equation (A2), the bracket of the integration term can be divided into the following forms:

$$\int_0^1 \frac{1}{\sqrt{1-x_1^2}}\cos(cx_1)\,\mathrm{d}x_1 - \int_0^1 \sqrt{1-x_1^2}\cos(cx_1)\,\mathrm{d}x_1,$$

where $c = e\sqrt{1-y^2}$. Considering the integration formulation in reference [11]

$$\int_0^a (a^2-x^2)^{\beta-1}\cos(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\left(\frac{2a}{b}\right)^{\beta-1/2}\Gamma(\beta)J_{\beta-1/2}(ab)$$
$$a,\ \mathrm{Re}\beta > 0, \quad |\arg b| < \pi \tag{A3}$$

where $\Gamma(x)$ and $J_s(x)$ represent the Gamma function and the sth order Bessel's function respectively. Applying equation (A3) to equation (A2),

$$I = \left(\frac{ka}{\pi}\right)\left(\frac{kb}{\pi}\right)\int_0^1 (1-y^2)\left\{\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{1}{2}\right)J_0(c) - \frac{\sqrt{\pi}}{2}\frac{2}{c}\Gamma\left(\frac{3}{2}\right)J_1(c)\right\}\mathrm{d}y,$$

where

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}.$$

It is therefore obtained as

$$I = \left(\frac{ka}{\pi}\right)\left(\frac{kb}{\pi}\right)\frac{\pi}{2}\left[I_1 - I_2\right], \tag{A4}$$

where

$$I_1 = \int_0^1 (1 - y^2) J_0(e\sqrt{1 - y^2})\,\mathrm{d}y, \qquad I_2 = \frac{1}{e}\int_0^1 \sqrt{(1 - y^2} J_1(e\sqrt{1 - y^2}\,\mathrm{d}y.$$

For integration I_1 and I_2, let $y_1 = \sqrt{1 - y^2}$; then $\mathrm{d}y_1 = -(\sqrt{1 - y_1^2}/y_1)\,\mathrm{d}y$,

$$I_1 = \int_0^1 y_1^3 (1 - y_1^2)^{-1/2} J_0(ey_1)\,\mathrm{d}y_1, \qquad I_2 = \int_0^1 y_1^2 (1 - y_1^2)^{-1/2} J_1(ey_1)\,\mathrm{d}y. \tag{A5, A6}$$

Considering the integration formulation in reference [12]:

$$\int_0^a x^{0+2n+1}(a^2 - x^2)^{\beta-1} J_\upsilon(cx)\,\mathrm{d}x = \frac{1}{2}(\upsilon + 1)_n \Gamma(\beta) a^{2n+2\beta+\upsilon}\left(\frac{2}{ac}\right)^{n+\beta}$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{(\upsilon+1)_k}\binom{n}{k}\left(\frac{ac}{2}\right)^k J_{k+n+\beta+\upsilon}(ac)$$

$$(a, \operatorname{Re}\beta > 0, \operatorname{Re}\upsilon > -n - 1). \tag{A7}$$

Application of equation (A7) to equation (A5) gives

$$I_1 = \sqrt{\frac{2\pi}{e^3}}\left[J_{3/2}(e) - \frac{e}{2} J_{5/2}(e)\right]. \tag{A8}$$

Considering another integration formulation in reference [12]:

$$\int_0^a x^{\upsilon+1}(a^2 - x^2)^{\beta-1} J_\upsilon(cx)\,\mathrm{d}x = \frac{2^{\beta-1} a^{\beta+\upsilon}}{c^\beta}\Gamma(\beta) J_{\beta+\upsilon}(ac) \quad (a, \operatorname{Re}\beta > 0, \operatorname{Re}\upsilon > -1) \tag{A9}$$

Application of equation (A9) to equation (A6) gives

$$I_2 = \sqrt{\frac{\pi}{2e}} J_{3/2}(e). \tag{A10}$$

Equation (A4) can therefore be rewritten as

$$I = \left(\frac{ka}{\pi}\right)\left(\frac{kb}{\pi}\right)\frac{\pi}{2}\sqrt{\frac{\pi}{2e}}\left[\frac{1}{e} J_{3/2}(e) - J_{5/2}(e)\right], \tag{A11}$$

where

$$J_{3/2}(e) = \sqrt{\frac{2e}{\pi}}\left(\frac{\sin e}{e^2} - \frac{\cos e}{e}\right),$$

$$J_{5/2}(e) = \sqrt{\frac{2e}{\pi}}\left[\left(\frac{3}{e^3} - \frac{1}{e}\right)\sin e - \frac{3}{e^2} - \cos e\right].$$

Substitution of the above formulations into equation (A11) gives the following:

$$I = \frac{k^2 ab}{6\pi} 3\left[\frac{\sin e}{e} + \frac{2\cos e}{e^2} - \frac{2\sin e}{e^3}\right]. \tag{A12}$$

It is noted that when $e \to 0$, $I|_{e\to 0} = (k^2ab/6\pi)3[1 - 2/3] = k^2ab/6\pi$, it is the same as the volume of the ellipsoidal segment given in reference [5]. In addition, the volume of the two "slices" should be considered in the summation of $(c_0/c_{mn})\cos k_{mn}d$. First, the area of the two "slice", can be obtained as

$$S_x = \int_0^{ka/\pi} \sqrt{1 - \left(\frac{\pi m}{ka}\right)^2}\,\mathrm{d}m. \tag{A13}$$

Let $x_1\sqrt{1 - (\pi m/ka)^2}$. Then

$$S_x = \frac{ka}{\pi}\int_0^1 \frac{x_1^2 \cos(ex_1)}{\sqrt{1 - x_1^2}}\,\mathrm{d}x_1 = \frac{ka}{2}\left[J_0(e) - \frac{J_1(e)}{e}\right], \tag{A14}$$

where equation (A3) is used. Similarly,

$$S_y = \frac{kb}{2}\left[J_0(e) - \frac{J_1(e)}{e}\right]. \tag{A15}$$

Thus, the total volume representing the summation of $(c_0/c_{mn})\cos k_{mn}d$ can be written as

$$\begin{aligned} I_t &= I + S_x/2 + S_y/2 \\ &= \frac{k^2 ab}{2\pi}\left[\frac{\sin e}{e} + \frac{2\cos e}{e^2} - \frac{2\sin e}{e^3}\right] + \frac{k(a+b)}{4}\left[J_0(e) - \frac{J_1(e)}{e}\right], \end{aligned} \tag{A16}$$

where $\frac{1}{2}$ is the width of each "slice" (similar to the concept in reference [5]).